States and Observables in Semiclassical Field Theory: a Manifestly Covariant Approach

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Abstract

A manifestly covariant formulation of quantum field Maslov complex-WKB theory (semiclassical field theory) is investigated for the case of scalar field. The main object of the theory is "semiclassical bundle". Its base is the set of all classical states, fibers are Hilbert spaces of quantum states in the external field. Semiclassical Maslov states may be viewed as points or surfaces on the semiclassical bundle. Semiclassical analogs of QFT axioms are formulated. A relationship between covariant semiclassical field theory and Hamiltonian formulation is discussed. The constructions of axiomatic field theory (Schwinger sources, Bogoliubov S-matrix, Lehmann-Symanzik-Zimmermann R-functions) are used in constructing the covariant semiclassical theory. A new covariant formulation of classical field theory and semiclassical quantization proposal are discussed.

Keywords: Maslov semiclassical theory, axiomatic quantum field theory, Bogoliubov S-matrix, Lehmann-Symanzik-Zimmermann approach, Schwinger sources, Peierls brackets.

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1 Introduction

Semiclassical methods are very important for quantum mechanics and field theory. The most convenient and general semiclassical approach for quantum mechanics is the Maslov theory of Lagrangian manifolds with complex germs [1, 2]. It is based on the direct substitution of the hypothetical wave function to the Schrodinger equation, possesses the mathematically rigorous justification (estimation of the accuracy) and involves other semiclassical methods as partial cases: the Ehrenfest approach, the oscillator approximation, the wave packet and WKB methods, the Maslov canonical operator method [3].

When one generalizes the Maslov complex germ theory to the quantum field models, one should take into account the specific features of QFT such as relativistic Poincare invariance, divergences and renormalization. Direct application of the Maslov approach is possible in Hamiltonian formulation of QFT only: one should first construct the Hamiltonian of the theory from the Lagrangian, write the Schrodinger equation with the help of the canonical quantization technique and use the Maslov substitution. It is not easy to investigate divergences and Poincare invariance in within this framework.

Some results of the Hamiltonian semiclassical field theory are presented in section 2. The QFT system with the Lagrangian depending on the small parameter h ("Planck constant") as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{h}V(\sqrt{h}\varphi), \tag{1.1}$$

is considered. Here $V(\Phi) \sim \frac{m^2}{2}\Phi^2$ as $\Phi \to 0$. Then it is possible to write the corresponding Schrodinger equation. The Maslov complex-WKB (complex-germ) substitution is

$$\Psi^t = e^{\frac{i}{h}\tilde{S}^t} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}[\Pi^t(\mathbf{x})\hat{\varphi}(\mathbf{x}) - \Phi^t(\mathbf{x})\hat{\pi}(\mathbf{x})]} f^t \tag{1.2}$$

One can show that under certain conditions the ansatz (1.2) satisfies the Schrodinger equation as $h \to 0$; relations on $\tilde{S}, \Pi, \Phi, f$ can be written; in particular, Π and Φ should obey classical field equations.

In addition to semiclassical evolution, it is possible to construct semiclassical Poincare transformations, field operators and other structures.

Contrary to the Hamiltonian approach, an axiomatic quantum field theory [4] is based on the most general principles, such as Poincare invariance, unitarity and causality. Examples are Lehmann-Symanzik-Zimmermann (LSZ) [5], Bogoliubov [6] and Bogoliubov-Medvedev-Polivanov [7] theories. The axiomatic perturbation theory is manifestly covariant. This simplifies an analysis of divergences and renormalization in high orders of perturbation theory [6, 8, 9].

Subsection 3.1 deals with semiclassical approximation in the axiomatic theory. Instead of states (1.2) (with non-covariant decomposition of space and time) one investigates "regularized" semiclassical states of the form

$$\Psi = e^{\frac{i}{h}\overline{S}} Texp[\frac{i}{\sqrt{h}} \int dx J(x) \hat{\varphi}_h(x)] \overline{f} \equiv e^{\frac{i}{h}\overline{S}} T_J^h \overline{f}, \tag{1.3}$$

Here $\hat{\varphi}_h(x)$ is a Heisenberg field operator; $J(x)$ is a c-number function (Schwinger classical source, cf.[10]). It is supposed to vanish, except for the case $x^0 \in [T_-, T_+]$. $Texp$ is a T-exponent.

It happens that the following property is satisfied in the leading order in h as $t > T_+$:

$$\hat{\varphi}_h(x) T_J^h \overline{f} \simeq \frac{1}{\sqrt{h}} \overline{\Phi}(x) T_J^h \overline{f}, \tag{1.4}$$

Here the function $\overline{\Phi}(x) \equiv \overline{\Phi}_J(x)$ is a solution of the Cauchy problem

$$\partial_\mu \partial^\mu \overline{\Phi} + V'(\overline{\Phi}) = J, \qquad \overline{\Phi}|_{t<T_-} = 0. \tag{1.5}$$

It is a classical field generated by the source J.

It is shown in section 4 that the state (1.3) approximately equals to (1.2) under certain conditions on $(\overline{S}, J, \overline{f})$ and $(\tilde{S}, \Pi, \Phi, f)$. For the important partial case $T_+ < 0$, the functions Φ and Π coincides with classical field $\overline{\Phi}$ and its time derivative as $t = 0$ correspondingly.

There is the following specific feature of the covariant approach to semiclassical field theory. Different sources $J_1 \neq J_2$ may generate the same semiclassical state (1.3), provided that the fields $\overline{\Phi}_{J_1}$ and $\overline{\Phi}_{J_2}$ generated by them coincide at $t > T_+$. Therefore, one should take into account the equivalence relation between semiclassical states analogously to gauge invariance in gauge theories. This property is considered in subsection 3.2. In particular, for the fields $\overline{\Phi}_J$ with compact support, one has

$$T_J^h \overline{f} \simeq e^{\frac{i}{h}\overline{I}} W \overline{f} \tag{1.6}$$

for some phase $\overline{I}$ and operator W being an S-matrix in the classical background field $\overline{\Phi}$.

Starting from relation (1.6), one can derive *from the first principles* classical equations of motion (stationary action principle) and commutation relations for semiclassical fields *without using the postulate of canonical quantization*. Therefore, one can investigate a relationship between axiomatic theory and Hamiltonian formalism. For operator W_J, one obtains analogs of Bogoliubov axioms of Poincare invariance, unitarity, causality.

2 Hamiltonian semiclassical field theory

2.1. Consider the Schrodinger equation for the model with Lagrangian (1.1):

$$i\frac{d}{dt}\Psi(t) = \int d\mathbf{x}\left[\frac{1}{2}\hat{\pi}^2(\mathbf{x}) + \frac{1}{2}(\nabla\hat{\varphi}(\mathbf{x}))^2 + \frac{1}{h}V(\sqrt{h}\hat{\varphi}(\mathbf{x}))\right]\Psi(t). \tag{2.1}$$

Here $\Psi(t)$ is a time-dependent state vector (element of the space $\mathcal{H}^h$), $\hat{\varphi}$ and $\hat{\pi}$ are field and momentum operators. They satisfy the well-known canonical commutation relations. In functional Schrodinger representation, states at fixed time moment are present as functionals $\Psi[\varphi(\cdot)]$, while operators $\hat{\varphi}$ and $\hat{\pi}$ are written as $\hat{\varphi}(\mathbf{x}) \equiv \varphi(\mathbf{x})$, $\hat{\pi}(\mathbf{x}) \equiv -i\frac{\delta}{\delta\varphi(\mathbf{x})}$. Semiclassical state (1.2) is presented as

$$\Psi[t,\varphi(\cdot)] = e^{\frac{i}{h}S^t} e^{\frac{i}{h}\int d\mathbf{x}\Pi^t(\mathbf{x})[\varphi(\mathbf{x})\sqrt{h}-\Phi^t(\mathbf{x})]} f^t[\varphi(\cdot) - \frac{\Phi^t(\cdot)}{\sqrt{h}}] \equiv (K^h_{S^t,\Pi^t,\Phi^t} f^t)[\varphi(\cdot)], \tag{2.2}$$

Here $S^t = \tilde{S}^t + \frac{1}{2}\int d\mathbf{x}\Pi^t(\mathbf{x})\Phi^t(\mathbf{x})$, f^t is an h-independent functional. Substituting functional (2.2) to the Schrodinger equation, one obtains the following relations [11, 12]:
() Π, Φ obey the classical equations:

$$\dot{\Phi}^t = \Pi^t, \qquad -\dot{\Pi}^t = -\Delta\Phi^t + V'(\Phi^t); \tag{2.3}$$

() S is a classical action, i.e.

$$\dot{S}^t = \int d\mathbf{x}[\Pi^t(\mathbf{x})\dot{\Phi}^t(\mathbf{x}) - \frac{1}{2}(\Pi^t(\mathbf{x}))^2 - \frac{1}{2}(\nabla\Phi^t(\mathbf{x}))^2 - V(\Phi^t(\mathbf{x}))]; \tag{2.4}$$

() f^t satisfies the Schrodinger equation in the external field $\Phi^t(\mathbf{x})$ with a quadratic Hamiltonian

$$\begin{gathered} i\dot{f}^t[\phi(\cdot)] = H_2^t f^t[\phi(\cdot)]. \\ H_2^t = \int d\mathbf{x}\left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(\mathbf{x})\delta\phi(\mathbf{x})} + \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}V''(\Phi^t(\mathbf{x}))\phi^2(\mathbf{x})\right] \end{gathered} \tag{2.5}$$

To eliminate quantum field divergences in the leading order of the semiclassical expansion, one should choose (generally, non-Fock) representation of the canonical commutation relations in the external field and add to the quadratic Hamiltonian an infinite c-number Φ-dependent counterterm [12, 13].

2.2. Thus, semiclassical states can be identified with pairs (X, f), where $X \equiv (S, \Pi, \Phi)$ are classical variables at fixed moment of time, f (quantum state in the external field) is an element of an X-dependent Hilbert space. Evolution is viewed as a set of transformations $u_t : X \mapsto u_t X$, $U_t(u_t X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_t X}$. From the mathematical point of view, set

of pairs (X,f) can be interpreted [14] as a vector bundle with a base $\{X\}$ and fibers $\mathcal{F}_X$ ("semiclassical bundle"). Evolution may be viewed as a 1-parametric automorphism group of the semiclassical bundle. The operator $K^h_X : \mathcal{F}_X \to \mathcal{H}^h$ taking a semiclassical state to quantum state (2.2) is called as *a canonical operator*.

Let $\mathcal{U}^h_t$ be evolution transformation in the "exact" theory. Then one writes:

$$\mathcal{U}^h_t K^h_X f \simeq K^h_{u_t X} U_t(u_t X \leftarrow X) f. \tag{2.6}$$

Relation (2.6) can be generalized to the case of transformation $\mathcal{U}^h_g$ corresponding to an element of the Poincare group $g \in G$:

$$\mathcal{U}^h_g K^f_X \simeq K^h_{u_g X} U_g(u_g X \leftarrow X) f. \tag{2.7}$$

where

$$u_g : X \mapsto u_g X, \qquad U_g(u_g X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_g X}. \tag{2.8}$$

Construction of renormalized transformations (2.8) and check of the group property are nontrivial in the Hamiltonian approach [13].

Formulate analogs of QFT axioms for the Hamiltonian semiclassical field theory.

A1. *A semiclassical bundle is defined. The space of the bundle is a set of states of the semiclassical theory. The base $\mathcal{X}$ of the bundle is a set of classical states, the fibers $\mathcal{F}_X$ are Hilbert state spaces of a quantum system in the external field $X \in \mathcal{X}$.*

2 (relativistic invariance). *The Poincare group G acts on the semiclassical bundle: to each $g \in G$ an automorphism (2.8) of the semiclassical bundle is assigned; the group property is satisfied:*

$$u_{g_1 g_2} = u_{g_1} u_{g_2},$$
$$U_{g_1 g_2}(u_{g_1 g_2} X \leftarrow X) = U_{g_1}(u_{g_1 g_2} X \leftarrow u_{g_2} X) U_{g_2}(u_{g_2} X \leftarrow X).$$

Investigate an analog of the field axiom. Since at $t = 0$

$$\hat{\varphi}(\mathbf{x}) K^h_X f = K^h_X \left[\frac{\Phi(\mathbf{x})}{\sqrt{h}} + \hat{\phi}(\mathbf{x}) \right] f,$$

where $\hat{\phi}(\mathbf{x})$ is a multiplicator by $\phi(\mathbf{x})$ in the functional Schrodinger representation, for the Heisenberg field operator one has:

$$\begin{gathered} \hat{\varphi}_h(\mathbf{x},t) K^h_X f = \mathcal{U}^h_{-t} \hat{\varphi}(\mathbf{x}) \mathcal{U}^h_t K^h_X f \simeq \mathcal{U}^h_{-t} \hat{\varphi}(\mathbf{x}) K^h_{u_t X} U_t(u_t X \leftarrow X) f = \\ \mathcal{U}^h_{-t} K^h_{u_t X} \left[\tfrac{\Phi^t(\mathbf{x})}{\sqrt{h}} + \hat{\phi}(\mathbf{x}) U_t(u_t X \leftarrow X) \right] f = \\ K^h_X \left[\tfrac{\Phi^t(\mathbf{x})}{\sqrt{h}} + U_{-t}(X \leftarrow u_t X) \hat{\phi}(\mathbf{x}) U_t(u_t X \leftarrow X) \right] f. \end{gathered} \tag{2.9}$$

Therefore, one has

$$\hat{\varphi}_h(x) K^h_X f \simeq K^h_X \left[\frac{\Phi(x|X)}{\sqrt{h}} + \hat{\phi}(x|X) \right] f, \tag{2.10}$$

Here $\Phi(x|X)$ is a c-number function. It depends on the space-time argument x and classical state X. $\hat{\phi}(x|X)$ is an operator-valued distribution in $\mathcal{F}_X$. The following equations are satisfied:

$$\partial_\mu \partial^\mu \Phi(x|X) + V'(\Phi(x|X)) = 0, \qquad \partial_\mu \partial^\mu \hat{\phi}(x|X) + V''(\Phi(x|X))\hat{\phi}(x|X) = 0.$$

They can be obtained from the Heisenberg equations of motion for $\hat{\varphi}_h(x)$.

Consider the property of Poincare invariance of fields in the semiclassical approximation. In the "exact" theory, for Poincare transformation $g = (a, \Lambda)$ of the form $x' = \Lambda x + a$, one has

$$\mathcal{U}^h_{g^{-1}} \hat{\varphi}_h(x) \mathcal{U}^h_g = \hat{\varphi}_h(w_g x), \qquad w_g x = \Lambda^{-1}(x - a). \tag{2.11}$$

Analogously to eq.(2.9), one obtains that

$$\hat{\varphi}_h(w_g x) K^h_X f \simeq K^h_X \left[\frac{\Phi(w_g x|X)}{\sqrt{h}} + \hat{\phi}(w_g x|X) \right] f.$$

One comes to the following semiclassical analogs of field axioms.

3 (field operator). *To each semiclassical state $X \in \mathcal{X}$ a c-number function $\Phi(x|X)$ and an operator-valued distribution $\hat{\phi}(x|X)$ in $\mathcal{F}_X$ are assigned.*

A4 (relativistic covariance of fields). *The following properties are satisfied:*

$$\begin{gathered} \Phi(w_g x|X) = \Phi(x|u_g X); \\ \hat{\phi}(w_g x|X) = U_{g^{-1}}(X \leftarrow u_g X)\hat{\phi}(x|u_g X) U_g(u_g X \leftarrow X). \end{gathered} \tag{2.12}$$

2.3. One can also introduce other structures [14] on the semiclassical bundle. They are related with semiclassical approximation rather than fields or Poincare invariance.

Let us shift a classical state $X \in \mathcal{X}$ by a small quantity $h\delta X$ of the order h. Then the quantum state $K^h_X f$ will be multiplied by a c-number. Denote it as $e^{-i\omega_X[\delta X]}$:

$$K^h_{X+\delta X} f \simeq e^{-i\omega_X[\delta X]} K^h_X f. \tag{2.13}$$

If one shifts a classical state by a quantity of the order $\sqrt{h}$, transformation of the canonical operator becomes more complicated:

$$K^h_{X+\sqrt{h}\delta X} \simeq const K^h_X e^{i\Omega_X[\delta X]} f. \tag{2.14}$$

Here *const* is a c-number multiplicator, $\Omega_X[\delta X]$ is a Hermitian operator. Moreover, ω_X and Ω_X are c-number and operator-valued 1-forms on $\mathcal{X}$ correspondingly. It happens that under some general requirements the commutator of operators $\Omega_X[\delta X]$ is a c-number. The c-number is related with 2-form $d\omega_X$ (differential of the 1-form ω_X):

$$[\Omega_X[\delta X];\Omega_X[\delta X']] = -id\omega_X(\delta X, \delta X'). \tag{2.15}$$

For our case,

$$\omega_X[\delta X] = \int d\mathbf{x}\Pi(\mathbf{x})\delta\Phi(\mathbf{x}) - \delta S, \tag{2.16}$$

$$\Omega_X[\delta X] = \int d\mathbf{x}[\delta\Pi(\mathbf{x})\hat{\varphi}(\mathbf{x}) - \delta\Phi(\mathbf{x})\hat{\pi}(\mathbf{x})]. \tag{2.17}$$

An important property of 1-forms ω and Ω is their independence of symmetry transformations. This property means that quantum state vectors $K^h_{X^t+h\delta X^t}f^t$ and $K^h_{X^t+\sqrt{h}\delta X^t}f^t$ should approximately satisfy the Schrodinger equation, provided that δX^t is a solution to the variation system. One comes to the following axioms.

5. *The 1-forms ω ($\omega_X[\delta X] \in \mathbf{R}$) and Ω ($\omega_X[\delta X]$ is an operator in $\mathcal{F}_X$) on $\mathcal{X}$ are given. The commutation relation (2.15) is satisfied.*

6 (relativistic invariance of 1-forms). *Let $u_g(X+\delta X) \simeq X' + \delta X'$. Then*

$$\omega_{X'}[\delta X'] = \omega_X[\delta X]. \tag{2.18}$$

$$\Omega_{X'}[\delta X']U_g(X' \leftarrow X) = U_g(X' \leftarrow X)\Omega_X[\delta X]. \tag{2.19}$$

2.4. The Maslov theory of Lagrangian manifolds of complex germ [1, 2] allows us to construct semiclassical solutions of the Schrodinger equations that differs from (2.2). One of possible formulations is as follows [11, 14, 15]. One considers the states being superpositions of (2.2):

$$\int d\alpha K^h_{X(\alpha)} f(\alpha). \tag{2.20}$$

Here $\alpha = (\alpha_1, ..., \alpha_k)$, $(X(\alpha), f(\alpha))$ is a k-dimensional surface in a space of the semiclassical bundle. It happens that state (2.20) is of an exponentially small as $h \to 0$ norm, except for the special case

$$\omega_X[\frac{\partial X}{\partial \alpha_a}] = 0. \tag{2.21}$$

It is the only case to be considered. Under condition (2.21) ("the Maslov isotropic condition") the square of the norm of (2.20) is

$$h^{k/2} \int d\alpha (f(\alpha), \prod_a 2\pi\delta(\Omega_X[\frac{\partial X}{\partial \alpha_a}])f(\alpha)). \tag{2.22}$$

Therefore, for normalization it is necessary to multiply state (2.20) by $h^{-k/4}$. The delta-functions entering to (2.22) commute because of relations (2.15) and (2.21).

Note that formula (2.22) resembles the inner product that arises in one of the approaches to quantize the constrained systems [16].

Under Poincare transformations, state (2.20) is taken to

$$h^{-k/4}\int d\alpha K^h_{X(\alpha)} f(\alpha) \mapsto h^{-k/4}\int d\alpha K^h_{u_g X(\alpha)} U_g(u_g X(\alpha) \leftarrow X(\alpha)) f(\alpha).$$

Invariance of 1-forms ω and Ω is a main requirement of self-consistence of the Maslov theory of Lagrangian manifolds with complex germs since the Maslov isotropic condition (2.21), as well as the inner product (2.22) should conserve under Poincare transformations.

2.5. The covariant formulation of semiclassical field theory to be considered below resembles the semiclassical mechanics of constrained systems [17]. A specific features of both theories is that some of semiclassical states are equivalent. Namely, for any pair of classically equivalent states $X \sim X'$ an isomorphism of fibers $V(X' \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{X'}$ is specified. The following property should be satisfied:

$$V(X'' \leftarrow X) = X(X'' \leftarrow X')V(X' \leftarrow X). \tag{2.23}$$

Semiclassical states $(X, f) \sim (X', V(X' \leftarrow X)f)$ are called gauge-equivalent.

Semiclassical Poincare transformations, 1-forms ω and Ω, functions Φ and $\hat{\phi}$ should conserve the gauge equivalence relation. Therefore,

$$\begin{gathered} u_g X \sim u_g X', \\ V(u_g X' \leftarrow u_g X) U_g(u_g X \leftarrow X) = U_g(u_g X' \leftarrow X') V(X' \leftarrow X); \end{gathered} \quad \text{as } X \sim X' \tag{2.24}$$

$$\begin{gathered} \Phi(x|X) = \Phi(x|X'), \\ V(X' \leftarrow X)\hat{\phi}(x|X) = \hat{\phi}(x|X')V(X' \leftarrow X) \end{gathered} \quad \text{as } X \sim X' \tag{2.25}$$

$$\begin{gathered} \omega_X[\delta X] = \omega_{X'}[\delta X'], \\ V(X' \leftarrow X)\Omega_X[\delta X] = \Omega_{X'}[\delta X']V(X' \leftarrow X). \end{gathered} \quad \text{as} \quad \begin{gathered} X \sim X', \\ X + \delta X \sim X' + \delta X'. \end{gathered} \tag{2.26}$$

In particular, under infinitesimal gauge transformation $X + \delta X \sim X$ the relations $\omega_X[\delta X] = 0$, $\Omega_X[\delta X] = 0$ should be satisfied. Requirements (2.23), (2.24), (2.25), (2.26) are related with self-consistence of semiclassical theory.

3 An axiomatic approach to semiclassical field theory

3.1 Semiclassical states

Consider an axiomatic approach to construct the semiclassical field theory. We will not use equations of motion; instead, the usual requirements [4] of the axiomatic theory will be used. These are Poincare invariance, existence of vacuum state $|0>$, relativistic invariance of the field $\hat{\varphi}_h(x)$ and T-exponent

$$T_J^h = Texp\{\frac{i}{\sqrt{h}}\int dx J(x)\hat{\varphi}_h(x)\}. \tag{3.1}$$

Other requirements will be also written explicitly.

Requirement 1. *Hilbert spaces $\mathcal{H}^h$ coincide, while operators $\hat{\varphi}_h(x)$ and $\mathcal{U}_g^h$ are expanded into asymptotic series in $\sqrt{h}$.*

Requirement 1 is usual, for example, for the S-matrix approach in the asymptotic in-representation [4, 8].

Denote these operators in the leading order in h as $\hat{\varphi}_0(x)$ and U_g, while the corresponding Hilbert space will be denoted as $\mathcal{H}^0$.

Consider the semiclassical states of the form (1.3):

$$\Psi = e^{\frac{i}{h}\overline{S}}T_J^h\overline{f} \equiv \overline{K}^h_{\overline{S},J}\overline{f}, \tag{3.2}$$

Here $J(x)$ is a function with a compact support in the domain $x^0 \in [T_-, T_+]$. Identify states (3.2) with points on the semiclassical bundle. The base is a set of pairs $\overline{X} = \{\overline{S}, J(x)\}$. The fibers are Hilbert spaces $\mathcal{H}$. Investigate the axioms of semiclassical field theory formulated in the previous section.

2: Since $\mathcal{U}_g^h T_J^h\overline{f} = T^h_{u_gJ}\mathcal{U}_g^h\overline{f} \simeq T^h_{u_gJ}U_g\overline{f}$, $u_gJ(x) = J(w_gx)$, w_g has the form (2.11), the Poincare group acts on the semiclassical bundle as follows: $(\overline{S}, J, \overline{f}) \mapsto (\overline{S}, u_gJ, U_g\overline{f})$.

3: Investigate the field operator as $x^0 > T_+$. One can write $\hat{\varphi}(x)T_J^h = \frac{\sqrt{h}}{i}\frac{\delta T_J^h}{\delta J(x)}$ and

$$\hat{\varphi}_h(x)T_J^h\overline{f} = T_J^h\frac{1}{\sqrt{h}}R(x|J)\overline{f}, \tag{3.3}$$

where

$$R(x|J) = -ih(T_J^h)^+\frac{\delta T_J^h}{\delta J(x)} \tag{3.4}$$

is a well-known LSZ R-function [4, 5]. Impose the following requirement on it.

Requirement 2. *R-function is expanded into an asymptotic series in $\sqrt{h}$:*

$$R(x|J) = \overline{\Phi}(x|J) + \sqrt{h}R^{(1)}(x|J) + ...$$

The leading order $\overline{\Phi}(x|J)$ is a c-number.

Under this requirement, the c-number function $\Phi(x|\overline{X})$ and the operator-valued distribution $\hat{\phi}(x|\overline{X})$ have the following form as $x^0 > T_+$

$$\Phi(x|\overline{X}) = \overline{\Phi}(x|J), \quad \hat{\phi}(x|\overline{X}) = R^{(1)}(x|J), \quad x^0 > T_+. \tag{3.5}$$

5: 1-forms ω and Ω can be introduced according to ref.[18]:

$$ih\delta\{e^{\frac{i}{h}\overline{S}}T_J^h\} \simeq T_J^h\{\overline{\omega}_{\overline{X}}[\delta\overline{X}] - \sqrt{h}\overline{\Omega}_{\overline{X}}[\delta\overline{X}]\};$$

therefore, $\overline{\omega}_{\overline{X}}[\delta\overline{X}] - \sqrt{h}\overline{\Omega}_{\overline{X}}[\delta\overline{X}] \simeq -\int dx R(x|J)\delta J(x) - \delta\overline{S}$ and

$$\begin{array}{c} \overline{\omega}_{\overline{X}}[\delta\overline{X}] = -\int dx\overline{\Phi}(x|J)\delta J(x) - \delta\overline{S}; \\ \overline{\Omega}_{\overline{X}}[\delta\overline{X}] = \int dx R^{(1)}(x|J)\delta J(x). \end{array} \tag{3.6}$$

Thus, the main objects of the semiclassical field theory are presented via the LSZ R-function. The following properties of the R-function are well-known [4].

Proposition 1. *() $R(x|J)$ is a Hermitian operator; it depends only on $J(y)$ at $y^0 < x^0$ (Bogoliubov causality condition); as $x^0 < T_-$, it has the form $R(x|J) = \hat{\varphi}_h(x)\sqrt{h}$.*
() R-function is Poincare invariant: $\mathcal{U}_{g^{-1}}^h R(x|u_gJ)\mathcal{U}_g^h = R(w_gx|J)$;
() the following commutation relation is satisfied:

$$[R(x|J); R(y|J)] = -ih\left(\frac{\delta R(x|J)}{\delta J(y)} - \frac{\delta R(y|J)}{\delta J(x)}\right). \tag{3.7}$$

Corollary. *1. The function $\overline{\Phi}(x|J)$ is real, vanishes as $x^0 < T_-$, depends only on $J(y)$ at $y^0 < x^0$ and satisfies the Poincare invariance property $\overline{\Phi}(w_gx|J) = \overline{\Phi}(x|u_gJ)$.*
2. The operator distribution $R^{(1)}(x|J)$ is Hermitian, coincides with $\hat{\varphi}_0(x)$ $x^0 < T_-$, depends only on $J(y)$ at $y^0 < x^0$, satisfies the relativistic invariance property

$$U_{g^{-1}}R^{(1)}(x|u_gJ)U_g = R^{(1)}(w_gx|J) \tag{3.8}$$

and commutation relation

$$[R^{(1)}(x|J); R^{(1)}(y|J)] = -i\left(\frac{\delta\overline{\Phi}(x|J)}{\delta J(y)} - \frac{\delta\overline{\Phi}(y|J)}{\delta J(x)}\right). \tag{3.9}$$

Thus, relativistic invariance of fields and 1-forms is a corollary of general properties of the LSZ R-functions. The commutation relation (2.5) coincides with (2.15).

3.2 Equivalence of semiclassical states

It has been written in section 1 that some semiclassical states may be equivalent each other in the covariant approach. Say that $J \sim 0$ iff

$$T_J^h \overline{f} \simeq e^{\frac{i}{h}\overline{I}_J} W_J \overline{f} \tag{3.10}$$

for some c-number $\overline{I}_J$ and operator W_J being an asymptotic series in $\sqrt{h}$.

Denote by W_J^0 the operator W_J in the leading order in $\sqrt{h}$. Investigate the properties of $\overline{I}_J$ and W_J.

Proposition 2. *1. The operator W_J is unitary.*
2. Let $J \sim 0$. Then $u_g J \sim 0$ and

$$\overline{I}_{u_g J} = \overline{I}_J, \qquad U_g W_J^0 U_g^{-1} = W_{u_g J}^0. \tag{3.11}$$

3. Under condition $x \gtrsim suppJ$ the following properties are satisfied:

$$\overline{\Phi}(x|J) = 0, \quad R^{(1)}(x|J) = (W_J^0)^+ \hat{\varphi}_0(x) W_J^0. \tag{3.12}$$

The first property is a corollary of unitarity of the operator T_J^h. The second one can be obtained by applying the operator $\mathcal{U}_g^h$ to the relation (3.10). The properties (3.12) are corollaries of the relation $T_J^h R(x|J)\overline{f} = \sqrt{h}\hat{\varphi}_h(x) T_J^h \overline{f}, \qquad x \gtrsim suppJ$. It is taken to the form $R(x|J) = \sqrt{h} W_J^+ \hat{\varphi}(x) W_J$ and expanded into a perturbation series.

Proposition 3. *Under conditions $J \sim 0$, $J + \delta J \sim 0$*

$$\delta \overline{I} = \int dx \overline{\Phi}(x|J)\delta J(x), \qquad \int dx R^{(1)}(x|J)\delta J(x) = 0. \tag{3.13}$$

To check eq.(3.13), consider the variation of relation (3.10):
$\delta T_J^h \cdot \overline{f} \simeq \frac{i}{h}\delta\overline{I}_J T_J^h \overline{f} + T_J^h W_J^+ \delta W_J \overline{f}$. Therefore,

$$-ih(T_J^h)^+ \delta T_J^h = \delta \overline{I}_J - ih W_J^+ \delta W_J. \tag{3.14}$$

On the other hand, it follows from eq.(3.4) that

$$-ih(T_J^h)^+ \delta T_J^h = \int dx R(x|J)\delta J(x). \tag{3.15}$$

Comparing eqs.(3.14) and (3.15), we obtain relation (3.13).

The following statement is a corollary of the causality principle for W^0_J.

Proposition 4. *Let $J+\Delta J_2 \sim 0$, $J+\Delta J_1+\Delta J_2 \sim 0$, $supp\Delta J_2 \gtrsim supp\Delta J_1$. Then the operator $(W^0_{J+\Delta J_2})^+ W^0_{J+\Delta J_1+\Delta J_2}$ and c-number $-\overline{I}_{J+\Delta J_2} + \overline{I}_{J+\Delta J_1+\Delta J_2}$ do not depend on ΔJ_2.*

To check the proposition, it is sufficient to consider the operator $(T^h_{J+\Delta J_2})^+ T^h_{J+\Delta J_1+\Delta J_2} \simeq e^{\frac{i}{h}[-\overline{I}_{J+\Delta J_2}+\overline{I}_{J+\Delta J_1+\Delta J_2}]}(W^0_{J+\Delta J_2})^+ W^0_{J+\Delta J_1+\Delta J_2}$. It is ΔJ_2-independent.

Introduce an equivalence relation on the semiclassical bundle. Say that $J_1 \sim J_2$ iff the properties $J_1+J_+ \sim 0$, $J_2+J_+\sim 0$ are satisfied for some source J_+ with $suppJ_+ \gtrsim suppJ_1 \cup suppJ_2$

Proposition 5. *Let $J_1 \sim J_2$. Let also the function J'_+ satisfy the properties $suppJ'_+ \gtrsim suppJ_1 \cup suppJ_2$ $J_1+J'_+\sim 0$. Then $J_2+J'_+\sim 0$.*

To check the proposition, it is sufficient to use the property $T^h_{J_2+J'_+} = T^h_{J_1+J'_+}(T^h_{J_2+J_+})^+ T^h_{J_1+J_+}$.

Corollary. *Let $J_1\sim J_2$ and $J_2 \sim J_3$. Then $J_1\sim J_3$.*

Proposition 6. *Let $J_1 \sim J_2$. Then the relation*

$$e^{\frac{i}{h}\overline{S}_1} T^h_{J_1}\overline{f}_1 \simeq e^{\frac{i}{h}\overline{S}_2} T^h_{J_2}\overline{f}_2 \tag{3.16}$$

is satisfied iff

$$\overline{S}_1 + \overline{I}_{J_1+J_+} = \overline{S}_2 + \overline{I}_{J_2+J_+}, \qquad \overline{f}_2 = (W^0_{J_2+J_+})^+ W^0_{J_1+J_+}\overline{f}_1. \tag{3.17}$$

To check the proposition, one should apply the operator $T^h_{J_+}$ to sides of the relation (3.16).

Say that two semiclassical states are equivalent, $(\overline{X}_1,\overline{f}_1)\sim(\overline{X}_2,\overline{f}_2)$, iff relation (3.17) is satisfied. Under our notations

$$V(\overline{X}_2 \leftarrow \overline{X}_1) = (W^0_{J_2+J_+})^+ W^0_{J_1+J_+} \equiv V(J_2\leftarrow J_1). \tag{3.18}$$

The causality property (proposition 4) implies that definition (3.18) does not depend on choice of the source J_+.

Let us check properties (2.23) – (2.26). Relation (2.23) is evident. Eq.(2.24) is a corollary of Poincare invariance (proposition 2). Property (2.26) for ω can be checked as follows. Let $(\overline{S}_1, J_1) \sim (\overline{S}_2, J_2)$, $(\overline{S}_1+\delta\overline{S}_1, J_1+\delta J_1) \sim (\overline{S}_2+\delta\overline{S}_2, J_2+\delta J_2)$. According to eq.(3.17), this means that

$$\delta\overline{S}_1 + \int dx \overline{\Phi}(x|J_1)\delta J_1 = \delta\overline{S}_2 + \int dx\overline{\Phi}(x|J_2)\delta J_2,$$

or $\omega_{\overline{X}_1}[\delta\overline{X}_1]\omega_{\overline{X}_2}[\delta\overline{X}_2]$.

Let us now apply the operator $\hat{\varphi}_h(x)$ to equality (3.16). One finds that $e^{\frac{i}{h}\overline{S}_1}T^h_{J_1}R(x|J_1)\overline{f}_1 = e^{\frac{i}{h}\overline{S}_2}T^h_{J_2}R(x|J_2)\overline{f}_2$. Therefore,

$$R(x|J_2)V(J_2 \leftarrow J_1) = V(J_2 \leftarrow J_1)R(x|J_1).$$

Expand this relation into a perturbation series. For $x \gtrsim suppJ_1 \cup suppJ_2$, one finds:

$$\overline{\Phi}(x|J_1) = \overline{\Phi}(x|J_2), \quad R^{(1)}(x|J_2)V(J_2 \leftarrow J_1) = V(J_2 \leftarrow J_1)R^{(1)}(x|J_1). \tag{3.19}$$

Thus, relation (2.25) is checked at $x \gtrsim suppJ$ (Φ and $\hat{\phi}$ have been defined at these x only). Properties (2.25) can be used to extend the functions Φ and $\hat{\phi}$ and define them for other values of x. To do this, one should choose a source $J' \sim J$ such that $suppJ \lesssim x$ and set

$$\Phi(x|\overline{X}) \equiv \overline{\Phi}(x|J'); \quad \hat{\phi}(x|\overline{X}) \equiv V(J \leftarrow J')R^{(1)}(x|J')V(J' \leftarrow J).$$

Properties (2.25) are valid for the extensions of functions as well.

Check the relation (2.26) for Ω.

Proposition 7. *Let $J_1 \sim J_2$, $J_1 + \delta J_1 \sim J_2 + \delta J_2$. Then*

$$\int dx R^{(1)}(x|J_2)\delta J_2 V(J_2 \leftarrow J_1) = V(J_2 \leftarrow J_1)\int dx R^{(1)}(x|J_1)\delta J_1(x). \tag{3.20}$$

To check the proposition, introduce the following operator function

$$\tilde{R}(x|J) = T^h_J R(x|J)(T^h_J)^+. \tag{3.21}$$

It depends only on $J(y)$ at $y^0 > x^0$, obeys the boundary condition $\tilde{R}(x|J) = \hat{\varphi}(x)\sqrt{h}$ as $x \gtrsim suppJ$. For the case $J \sim 0$, the $\tilde{R}$-function has the form $\tilde{R}(x|J) = W_J R(x|J) W_J^+$, can be expand into an asymptotic series

$$\tilde{R}(x|J) = \tilde{\Phi}(x|J) + \sqrt{h}\tilde{R}^{(1)}(x|J) + ...,$$

with

$$\tilde{\Phi}(x|J) = \Phi(x|J), \qquad \tilde{R}^{(1)}(x|J) = W^0_J R^{(1)}(x|J)(W^0_J)^+. \tag{3.22}$$

Let $J_1 \sim J_2$ $J_1 + \delta J_1 \sim J_2 + \delta J_2$. This means that $J_1 + J_+ \sim 0$, $J_2 + J_+ \sim 0$, $J_1 + \delta J_1 + J_+ + \delta J_+ \sim 0$, $J_2 + \delta J_2 + J_+ + \delta J_+ \sim 0$ at $suppJ_+ \cup supp\delta J_+ \gtrsim suppJ_1 \cup suppJ_2 \cup supp\delta J_1 \cup supp\delta J_2$. It follows from eqs. (3.13), (3.18), (3.22), that relation (3.20) is equivalent to the following:

$$\begin{array}{c} (W^0_{J_2+J_+})^+ \int dx \tilde{R}^{(1)}(x|J_2 + J_+)\delta J_+(x) W^0_{J_1+J_+} = \\ (W^0_{J_2+J_+})^+ \int dx \tilde{R}^{(1)}(x|J_1 + J_+)\delta J_+(x) W^0_{J_1+J_+}, \end{array} \tag{3.23}$$

However, eq.(3.23) is a corollary of the causality condition for $\tilde{R}^{(1)}$: $\tilde{R}^{(1)}(x|J_1+J_+) = \tilde{R}^{(1)}(x|J_2+J_+) = \tilde{R}^{(1)}(x|J_+)$. Proposition is justified.

To obtain the classical field equations "from the first principles", impose also the following requirement.

Requirement 3. *For any field configuration $\Phi(x)$ with a compact support there exists a unique source $J \sim 0$ (denote it as $J = J(x|\Phi)$) generating the field configuration Φ: $\Phi(x) = \overline{\Phi}(x|J)$. It satisfies the locality condition*

$$\frac{\delta J(x|\Phi)}{\delta \Phi(y)} = 0, y \neq x. \tag{3.24}$$

Requrement 3 and eq.(3.13) imply the following statement.

Poposition 8. *1. The functional $I[\Phi] = \overline{I}_{J_\Phi} - \int dx J(x)\Phi(x)$ satisfies the locality property $I[\Phi] = \int dx \mathcal{L}(\Phi(x), \partial_\mu \Phi(x), ..., \partial_{\mu_1}...\partial\mu_n \Phi(x))$ and stationary action principle*

$$\frac{\delta I[\overline{\Phi}]}{\delta \Phi(x)} = -J(x). \tag{3.25}$$

2. The operator-valued distribution $R^{(1)}(x|J)$ satisfies the equation

$$\int dy \frac{\delta^2 I[\overline{\Phi}]}{\delta\Phi(x)\delta\Phi(y)} R^{(1)}(y|J) = 0. \tag{3.26}$$

In particular, the field $\hat{\varphi}_0(x)$ obeys the equation

$$\int dy \left. \frac{\delta^2 I[\overline{\Phi}]}{\delta\Phi(x)\delta\Phi(y)} \right|_{\Phi=0} \hat{\varphi}_0(y) = 0. \tag{3.27}$$

3. Denote by $D^{ret}_{\overline{\Phi}}(x,y)$ the retarded Green function of the problem

$$\int dy \frac{\delta^2 I[\overline{\Phi}]}{\delta\Phi(x)\delta\Phi(y)} \delta\Phi(y) = -\delta J(x), \qquad \delta\Phi|_{x^0<T_-} = 0.$$

It is defined from the relation $\delta\Phi(x) = \int dy D^{ret}_{\overline{\Phi}}(x,y)\delta J(y)$. Then

$$[R^{(1)}(x|J); R^{(1)}(y|J)] = -i(D^{ret}_{\overline{\Phi}}(x,y) - D^{ret}_{\overline{\Phi}}(y,x)). \tag{3.28}$$

In particular,

$$[\hat{\varphi}_0(x); \hat{\varphi}_0(y)] = -i(D^{ret}_0(x,y) - D^{ret}_0(y,x)). \tag{3.29}$$

The property (3.25) is another form of eq.(3.13), since $\delta I = \delta \overline{I} - \int dx (\delta J \overline{\Phi} + J \delta \overline{\Phi}) = - \int dx J \delta \overline{\Phi}$. The locality property for I is a corollary of (3.24). Moreover, let Φ and $\Phi + \delta\Phi$ be field configurations with compact support. Then for corresponding sources J, $J+\delta J$ one has $\delta J(y) = - \int dx \frac{\delta^2 I[\overline{\Phi}]}{\delta\Phi(x)\delta\Phi(y)} \delta\Phi(x)$. Therefore, eq.(3.13) is rewritten as $- \int dx dy R^{(1)}(y|J) \frac{\delta^2 I[\overline{\Phi}]}{\delta\Phi(x)\delta\Phi(y)} \delta\Phi(x) = 0$. One obtains then eq.(3.26) and its partial case (3.27). Commutation relation (3.28) is another form of (3.9). Proposition is justified.

Thus, the semiclassical theory is reconstructed in the leading order in h from the action functional $I[\Phi]$ *without postulates of the canonical quantization.* Namely, starting from $I[\Phi]$, one uniquely finds:

- equation of motion (3.25) for Φ; it allows us to find $\overline{\Phi}(x|J)$ from the boundary condition $\Phi|_{x^0<T_-} = 0$;

- equation of motion (3.27) and commutation relation (3.29) for free field $\hat{\varphi}_0(x)$;

- equation of motion (3.26) for the operators $R^{(1)}(x|J)$ and commutation relation (3.28) for these operators.

The right-hand side of the obtained relation (3.28) contains the Peierls bracket [19] (see also [20, 21]) of classical fields at points x and y. The Peierls bracket postulate was considered in [20] as a foundation of QFT.

If one fixes the representation for the operators $\hat{\varphi}_0$ (for example, the Fock representation) then

- relation $U_{g^{-1}} \hat{\varphi}_0(x) U_g = \hat{\varphi}_0(w_g x)$ allows us to reconstruct the operator U_g up to a c-number multiplier for the free theory; the multiplier is reconstructed fro, the condition $U_g|0>=|0>$;

- eq.(3.26) and initial condition $R^{(1)}|_{x^0<T_-} = \hat{\varphi}_0(x)$ allows us to reconstruct the operators $R^{(1)}(x|J)$ uniquely;

- relation (3.12) for $x^0 > T_+$ allows us to reconstruct the operator W_J^0 up to a c-number multiplier c_J; it follows from the causality condition that $\exp[i \int dx \mathcal{L}_1(\Phi(x), \partial_\mu \Phi(x), ..., \partial_{\mu_1} ... \partial\mu_n \Phi(x))]$, here $\mathcal{L}_1$ is a one-loop counterterm.

4 Correspondence of Hamiltonian and axiomatic approaches

4.1. We have considered Hamiltonian and axiomatic formulations of semiclassical field theory. Investigate their correspondence.

First of all, show that state (1.3) of the covariant approach can be taken to the form (2.2). Notice that the operator $T^h_J = Texp[\frac{i}{\sqrt{h}}\int dx J(x)\hat{\varphi}_h(x)]$ is related with the evolution transformation $U_J(0,t_-)$ for the system with Hamiltonian

$$H_J(t) = H - \frac{1}{\sqrt{h}}\int d\mathbf{x} J(\mathbf{x},t)\hat{\varphi}(\mathbf{x})$$

via the standard relation

$$T^h_J = U_J(0,t_-)e^{-iHt_-}, \qquad t_- < (suppJ)^0.$$

One also has

$$e^{-iHt_-}\overline{f} \simeq e^{-iH_0t_-}\overline{f}, \tag{4.1}$$

where H_0 is a Hamiltonian of the free field with mass m. The semiclassical state (4.1) is taken to (2.2) under semiclassical evolution with Hamiltonian H_J; one obtains the system of equations for S,Π,Φ,f; it is related to (2.3), (2.4), (2.5) by substitution $V(\Phi) \Rightarrow V(\Phi) - J\Phi$. Denote by $U_2(0,t_-)$ the evolution operator for eq.(2.5). Then

$$e^{\frac{i}{h}\overline{S}}T^h_J\overline{f} \simeq K^h_{S,\Pi,\Phi}f, \qquad (suppJ)^0 = [T_-,T_+] \subset (-\infty,0).$$

Here

$$\begin{array}{c} \Pi(\mathbf{x}) = \dot{\overline{\Phi}}(x|J), \quad \Phi(\mathbf{x}) = \overline{\Phi}(x|J), \quad S = \overline{S} + I_-[\Phi], \quad x^0 = 0 \\ I_-[\overline{\Phi}] = \int_{x^0<0} dx[\frac{1}{2}\partial_\mu\overline{\Phi}\partial^\mu\overline{\Phi} - V(\overline{\Phi}) + J\overline{\Phi}], \\ f = \mathcal{V}_{\overline{X}}\overline{f}, \quad \mathcal{V}_{\overline{X}} = U_2(0,t_-)e^{-iH_0t_-}, \quad t_- < T_-. \end{array} \tag{4.2}$$

Thus, the property (1.4) is obtained.

4.2. Investigate the correspondence of classical field theories in Hamiltonian and axiomatic aproaches. For the former formulation, an extended phase space (the base of the semiclassical bundle) consists of sets $(S,\Pi(\mathbf{x}),\Phi(\mathbf{x}))$; the 1-form ω is

$$\omega_X[\delta X] = \int d\mathbf{x}\Pi(\mathbf{x})\delta\Phi(\mathbf{x}) - \delta S. \tag{4.3}$$

For the latter formulation, an extended phase space may be viewed as a surface on the space of sets $\{\overline{X} = (S, J(x), \overline{\Phi}(x))\}$; the equation of surface has the form (3.25):

$$\partial_\mu\partial^\mu\overline{\Phi}(x) + V'(\overline{\Phi}(x)) = J(x), \quad \Phi|_{x^0<T_-} = 0. \tag{4.4}$$

The 1-form ω is written as

$$\overline{\omega}_X[\delta\overline{X}] = -\int dx\overline{\Phi}(x)\delta J(x) - \delta\overline{S}. \tag{4.5}$$

Relations (4.4) and (4.5) allows us to interpret $J(x)$ and $\overline{\Phi}(x)$ as momenta and coordinates correspondingly, while relation (4.4) is a set of first-class constraints.

Equivalence of Hamiltonian and covariant formulations of classical field theory is a corollary of relation $\overline{\omega}_{\overline{X}}[\delta\overline{X}] = \omega_X[\delta X]$. It can be checked by a direct calculation. One should use properties (4.2) and relation $\delta S = \delta\overline{S} + \delta I_-[\overline{\Phi}]$. The symplectic 2-forms $d\omega$ also coincide:

$$\int d\mathbf{x}[\delta\Pi_1(\mathbf{x})\delta\Phi_2(\mathbf{x}) - \delta\Pi_2(\mathbf{x})\delta\Phi_1(\mathbf{x})] = \int dx[\delta J_1(x)\delta\Phi_2(x) - \delta J_2(x)\delta\Phi_1(x)].$$

Note that Poincare transformation generators have the following form in the covariant formulation:

$$\begin{array}{c} \mathcal{P}^\mu = \int dx J(x)\partial^\mu\overline{\Phi}(x), \\ \mathcal{M}^{\mu\nu} = \int dx J(x)(x^\mu\partial^\nu - x^\nu\partial^\mu)\overline{\Phi}(x). \end{array} \tag{4.6}$$

Making use of eq.(4.4), one can take the generators to the form of the Hamiltonian field theory. For example, $\mathcal{P}^\mu$ is an integral of $T^{\mu 0}$ over space. Analogously to eq.(4.6), it is possible to write generators of any transformation that conserve the action. The generators coincide with Noether integrals of motion.

4.3. Investigate now the relationship between other objects of semiclassical field theory in Hamiltonian and axiomatic approaches: semiclassical Poincare transformations $U_g(u_g X \leftarrow X)$, semiclassical fields $\hat{\Phi}(x|X)$, 1-forms Ω. Equivalence of the approaches means that

$$\mathcal{V}_{u_g\overline{X}} U_g \overline{f} = U_g(u_g X \leftarrow X)\mathcal{V}_{\overline{X}}\overline{f}; \tag{4.7}$$

$$\mathcal{V}_{\overline{X}} R^{(1)}(x|J)\overline{f} = \hat{\phi}(x|X)\mathcal{V}_{\overline{X}}\overline{f}; \qquad x \underset{\sim}{>} suppJ; \tag{4.8}$$

$$\mathcal{V}_{\overline{X}} \int dx R^{(1)}(x|J)\delta J(x)\overline{f} = \int d\mathbf{x}[\delta\Pi(\mathbf{x})\phi(\mathbf{x}) - \delta\Phi(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta\phi(\mathbf{x})}]\mathcal{V}_{\overline{X}}\overline{f}; \tag{4.9}$$

Relation (4.9) can be checked as follows. First of all, take into account relation (4.4) and property

$$\partial_\mu\partial^\mu\delta\overline{\Phi} + V''(\overline{\Phi})\delta\overline{\Phi} = \delta J, \qquad \delta\overline{\Phi}|_{x \underset{\sim}{<} supp\delta J} = 0 \tag{4.10}$$

Take the operator in the left-hand side to the form:

$$\begin{array}{c} \int dx R^{(1)}(x|J)\delta J(x) = \int dx R^{(1)}(x|J)(\partial_\mu\partial^\mu\delta\overline{\Phi}(x) + V''(\overline{\Phi})\delta\overline{\Phi}(x)) = \\ \int dx[R^{(1)}(x|J)\partial_\mu\partial^\mu\delta\overline{\Phi}(x) - \delta\overline{\Phi}(x)\partial_\mu\partial^\mu R^{(1)}(x|J)] = \\ \int d\sigma^\mu[R^{(1)}(x|J)\partial^\mu\delta\overline{\Phi}(x) - \delta\overline{\Phi}(x)\partial^\mu R^{(1)}(x|J)] \end{array} \tag{4.11}$$

Integration in (4.11) is performed over and space-like surface such that $x \underset{\sim}{>} suppJ$.

By $\delta\Phi_+(x)$ we denote the solution of the problem

$$[\partial_\mu\partial^\mu + V''(\overline{\Phi}(x))]\delta\Phi_+ = 0, \qquad \delta\Phi_+|_{x^0>T_+} = \delta\overline{\Phi}; \tag{4.12}$$

then formula (4.11) is taken to the form

$$\int dx R^{(1)}(x|J)\delta J(x) = \int d\sigma^\mu [R^{(1)}(x|J)\partial^\mu \delta\Phi_+(x) - \delta\Phi_+(x)\partial^\mu R^{(1)}(x|J)], \tag{4.13}$$

or

$$\int dx R^{(1)}(x|J)\delta J(x) = \int d\sigma^\mu [\hat{\varphi}_0(x)\partial^\mu \delta\Phi_+(x) - \delta\Phi_+(x)\partial^\mu \hat{\varphi}_0(x)]. \tag{4.14}$$

The space-like surface of integration is arbitrary in eq.(4.13) and satisifes the condition $x \lesssim suppJ$ in eq.(4.14).

Notice that the operator $A(t) = \int_{x^0=t} d\mathbf{x}[\delta\dot{\Phi}_+(\mathbf{x},t)\hat{\varphi}(\mathbf{x}) - \delta\Phi_+(\mathbf{x},t)\hat{\pi}(\mathbf{x})]$ commutes with $i\frac{d}{dt} - H_2^t$ and takes the solutions of eq.(2.5) to the solutions. Therefore, the left-hand side of eq.(4.9) is as follows:

$$\mathcal{V}_{\overline{X}} e^{iH_0t_-} A(t_-) e^{-iH_0t_-}\overline{f} = U_2(0,t_-)A(t_-)e^{-iH_0t_-}\overline{f} = A(0)U_2(0,t_-)e^{-iH_0t_-}\overline{f} = A(0)\mathcal{V}_{\overline{X}}\overline{f}$$

It coincides with the right-hand side. Relation (4.9) is checked. Its left-hand side can be presented as an integral (4.11) over the surface $x^0 = 0$; therefore, for $x^0 = 0$

$$\mathcal{V}_{\overline{X}} R^{(1)}(x|J) = \hat{\phi}(x)\mathcal{V}_{\overline{X}}, \qquad \mathcal{V}_{\overline{X}}\partial_0 R^{(1)}(x|J) = \partial_0\hat{\phi}(x)\mathcal{V}_{\overline{X}},$$

Since $\hat{\phi}(x)$ and $R^{(1)}(x|J)$ satisfy the same second-order differential equation at $x \gtrsim suppJ$, one comes to relation (4.8).

Property (4.7) can be obtained from relativistic invariance of fields up to a c-number multiplier. Namely, the operators $U_g(u_gX \leftarrow X)$ are found from (2.12) up to a multiplier, while the operator $\mathcal{V}_{u_g\overline{X}} U_g \mathcal{V}_{\overline{X}}^{-1}$ satisifes (2.12). To investigate the c-number multiplier, one should analyze renormalization of Poincare transformations in more details.

Notice also that formulas (4.8) and (4.9) allows us to obtain useful relations for semi-classical field theory $\hat{\phi}(x|X)$ and 1-form Ω. For the simplicity, denote

$$\Omega\{\delta\Phi(\cdot)\} = \Omega(\delta\Phi, \delta\dot{\Phi}) = \int d\mathbf{x}[\delta\dot{\Phi}(\mathbf{x},0)\hat{\phi}(\mathbf{x}) - \delta\Phi(\mathbf{x},0)\hat{\pi}(\mathbf{x})],$$

iff $\delta\Phi$ is a solution of equation

$$\partial_\mu\partial^\mu\delta\Phi + V''(\Phi(x))\delta\Phi = 0.$$

It follows from (4.8), (4.9), (4.13) that

$$\int dx\hat{\phi}(x|X)\delta J(x) = \Omega\{\delta\Phi_+(\cdot)\}. \tag{4.15}$$

Here $\delta\Phi_+$ is found from relations (4.10) and (4.12).

Property (4.15) shows us that axioms of semiclassical field and its Poincare invariance are not independent. It is possible to define semiclassical field $\hat{\phi}(x|X)$ by eq.(4.15). Its Poincare invariance is a corollary of relativistic invariance of 1-form Ω.

5 Conclusions

Thus, it is possible to derive main formulas of classical and semiclassical field theroy (stationary action proinciple, equations of motion and commutation relations for semiclassical fields) from the first principles of axiomatic QFT and general requiremnets.

The main difficulty of axiomatic QFT is that there are no nontrivial model obeying all the axioms. Therefore, when one investigates models of semiclassical field theory, it seems to be reasonable to consider the properties that are essential in the leading order of h and check only them. These essential properties are axioms A1-A6 in Hamiltonian approach. For the covariant approach, one should additionally investigate properties (2.23) – (2.26).

To study semiclassical perturbation theory, one should generalize the considered axioms. It is also interesting to construct semiclassical gauge theories in the axiomatic approach. The authors is going to consider these problems in the following publications.

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